

## Reduction property and dimensional order property

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## REDUCTION PROPERTY AND DIMENSIONAL ORDER PROPERTY

By

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### 1. Introduction

Let  $T$  be a theory with a relational language  $L$  including a unary predicate  $P$ . Let  $M$  be a model of  $T$  and  $N$  the  $L^-$ -structure  $P^M = \{a \in |M| : M \models P(a)\}$  where  $L^- \subset L - \{P\}$ . The following question seems to be natural:

QUESTION. Which properties of  $T^- = Th(N)$  are also possessed by  $T$  (under certain conditions)?

There are a few papers treating the question. In [HP] Hodges and Pillay have shown that if  $T$  is minimal over  $P$  (definition 2.3) and every automorphism of  $N$  can be extended to an automorphism of  $M$  (they call  $M$  is a symmetric extension of  $N$ ), then  $N$  is  $\aleph_0$ -categorical iff  $M$  is  $\aleph_0$ -categorical. In [KT] Kikyo and Tsuboi defined the  $\emptyset$ -reduction property, the reduction property, the strong reduction property, and the uniform reduction property. These reduction properties are model theoretical rephrasing of symmetry. They have shown that if  $T$  is minimal over  $P$  and has the uniform reduction property (i.e., for each  $L$ -formula  $\varphi(\bar{x}\bar{y})$ , there is an  $L^-$ -formula  $\psi(\bar{x}\bar{z})$  such that  $(\forall \bar{y})(\exists \bar{z} \in P)(\forall \bar{x} \in P)[\varphi(\bar{x}\bar{y}) \leftrightarrow \psi^P(\bar{x}\bar{z})]$  holds), then  $T^-$  is  $\lambda$ -stable iff  $T$  is  $\lambda$ -stable and  $T^-$  is unidimensional iff  $T$  is unidimensional.

In this paper, we mainly deal with the  $\emptyset$ -reduction property (definition 2.1). The  $\emptyset$ -reduction property together with the minimality condition ensures that  $T$  is not far from  $T^-$  if  $T$  is stable. But the  $\emptyset$ -reduction property is not so strong for an unstable theory. In fact there is a theory  $T$  such that  $T$  has the  $\emptyset$ -reduction property over  $P$ , is minimal over  $P$ , and the number of models of  $T$  is more than that of  $T^-$ .

EXAMPLE. Let  $A$  be a model whose theory has uncountably many countable models. Let

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- (i)  $L = \{P, R\} \cup \text{the language of } A$ ,
- (ii)  $L^- = L - \{P\}$ ,
- (iii)  $M = \langle M; P^M, R^M, \text{the structure of } A \rangle$ ,
- (iv)  $M = A \cup B \cup C$ ,
- (v)  $P^M = B \cup C$ ,
- (vi)  $C$  is the set of all bijections from  $A$  to  $B$ ,
- (vii)  $R^M(x, y, f)$  iff  $x \in A \wedge y \in B \wedge f \in C \wedge f(x) = y$ .

Since  $(\forall xyz)\neg R(x, y, z)$  holds in  $N$ ,  $T = Th(M)$  has the  $\emptyset$ -reduction property over  $P$  and  $T^- = Th(N)$  has a unique countable model.  $T$  is minimal over  $P$  because each element of  $C$  is a bijection from  $A$  to  $B$ . But  $T$  has uncountably many countable models since  $Th(A)$  does.

When  $T$  is superstable, we can easily prove that if  $T$  has the  $\emptyset$ -reduction property over  $P$  and is minimal over  $P$ , then the number of  $a$ -models of  $T$  is equal to that of  $T^-$  (corollary 3.2). Furthermore, we prove the following:

**THEOREM (4.1, 5.1).** Let  $T$  be a superstable theory with the  $\emptyset$ -reduction property over  $P$ . If  $T$  is minimal over  $P$ , then

- (1)  $T$  has the DOP (dimensional order property) iff  $T^-$  has the DOP.
- (2)  $T$  is deep iff  $T^-$  is deep.

For a stable theory, the  $\emptyset$ -reduction property implies a reduction property for formulas with parameters (in [KT], it was called the reduction property). The key point of the proofs of the above theorems is that under this reduction property, a type of an element in  $N$  is determined by its " $L^-$ -reduction" of the type (see lemma 2.4).

## 2. Preliminaries

Let  $T$  be a theory with a relational language  $L$  including a unary predicate  $P$ . Let  $M$  be a model of  $T$  and  $N$  the  $L^-$ -structure  $P^M = \{a \in |M| : M \models P(a)\}$  where  $L^- \subset L - \{P\}$ . As usual, we work in the big model  $\mathcal{M}$  of  $T$ . We may assume that a model of  $T^- = Th(N)$  is an  $L^-$ -elementary substructure of  $P^\mathcal{M}$ . The character  $M$  will denote an elementary submodel of  $\mathcal{M}$  in  $T$  and the character  $N$  will denote the set  $P^M$  which is a model of  $T^-$ .  $M$  and  $N$  may have subscript. For notational convenience, we usually assume that if  $M$  and  $N$  have the same subscript then  $N$  is the restriction of  $M$  to  $P$ . For example,  $N_i$  for  $P^{M_i}$ .

We write  $\bar{a}, \bar{b}, \dots$  for finite tuples of elements of  $\mathcal{M}$  and  $\bar{x}, \bar{y}, \dots$  for finite tuples of variables. When  $\varphi$  is an  $L^-$ -formula, we write  $\varphi^P$  for the restriction of  $\varphi$ , that is, the formula obtained from  $\varphi$  by restricting all the variables to  $P$ . For

example, if  $\varphi(x) = (\forall y)(\exists z)\psi(xyz)$  and  $\phi$  is open, then  $\varphi^P(x) \equiv (\forall y)(P(y) \rightarrow (\exists z)(\psi(xyz) \wedge P(x) \wedge P(z)))$ . We write  $(\forall \bar{x} \in P)\varphi(\bar{x}\bar{y})$  to express the formula  $(\forall x_1 \cdots \forall x_n)[P(x_1) \wedge \cdots \wedge P(x_n) \rightarrow \varphi(\bar{x}\bar{y})]$  where  $\bar{x} = x_1 \cdots x_n$ .

DEFINITION 2.1 ([KT, definition 1]). (1) We say that  $M$  has the  $\emptyset$ -reduction property over  $N$  if every  $L(\emptyset)$ -definable relation on  $N$  is  $L^-(\emptyset)$ -definable in  $N$ , i.e., for any  $L(\emptyset)$ -formula  $\varphi(\bar{x})$ , there is an  $L^-(\emptyset)$ -formula  $\psi(\bar{x})$  such that  $M \models (\forall \bar{x} \in P)[\varphi(\bar{x}) \leftrightarrow \psi^P(\bar{x})]$ .

(2) We say that  $M$  has the reduction property over  $N$  if every  $L(M)$ -definable relation on  $N$  is  $L^-(N)$ -definable in  $N$ .

If some model of  $T$  has the  $\emptyset$ -reduction property over  $N$ , then every model of  $T$  has the property. So we say that  $T$  has the  $\emptyset$ -reduction property over  $P$  if some model of  $T$  has this property.

The following lemma was used in [KT] without proof. For the sake of completeness, we prove it here.

LEMMA 2.2 ([KT, pp. 902]). If  $T$  is stable and has the  $\emptyset$ -reduction property over  $P$ , then every model  $M$  of  $T$  has the reduction property over  $N$ .

PROOF. Let  $M$  be a model of  $T$ ,  $\varphi(\bar{x}\bar{y})$  an  $L$ -formula, and  $\bar{a} \in M$ . We want to find an  $L^-$ -formula  $\psi(\bar{x}\bar{z})$  and a tuple  $\bar{b} \in N$  such that  $M \models (\forall \bar{x} \in P)[\varphi(\bar{x}\bar{a}) \leftrightarrow \psi^P(\bar{x}\bar{b})]$ . For this, it is sufficient to show that  $M \models [\varphi(\bar{c}\bar{a}) \leftrightarrow \psi^P(\bar{c}\bar{b})]$  for every  $\bar{c} \in N$ . By the stability, there are an  $L$ -formula  $\varphi'(\bar{x}\bar{z})$  and a tuple  $\bar{b} \in N$  such that  $M \models \varphi(\bar{c}\bar{a}) \leftrightarrow \varphi'(\bar{c}\bar{b})$  for every  $\bar{c} \in N$ . By the  $\emptyset$ -reduction property, there is an  $L^-$ -formula  $\psi(\bar{x}\bar{z})$  such that  $M \models (\forall \bar{x}\bar{z} \in P)[\varphi'(\bar{x}\bar{z}) \leftrightarrow \psi^P(\bar{x}\bar{z})]$ .  $\square$

DEFINITION 2.3 ([KT, definition 1]). We say that  $T$  is minimal over  $P$  if every model  $M$  of  $T$  is a minimal model over  $N$ .

We write  $tp^-(\bar{a}/B)$  for the  $L^-$ -type of  $\bar{a} \in N$  over  $B \subset N$ .

LEMMA 2.4. Let  $\bar{a}$  and  $\bar{b}$  be tuples from  $P^M$ .

(1)  $tp(\bar{a}/M)$  does not fork over  $N$ .

(2) If  $M$  has the reduction property over  $N$ , then  $tp(\bar{a}/M) = tp(\bar{b}/M)$  iff  $tp^-(\bar{a}/N) = tp^-(\bar{b}/N)$ .

(3) If  $M$  has the reduction property over  $N$ , then  $tp(\bar{a}/M\bar{b})$  does not fork over  $M$  iff  $tp^-(\bar{a}/N\bar{b})$  does not fork over  $N$  in the sense of  $T^-$ .

(4) If  $T$  has the  $\emptyset$ -reduction property over  $P$ , then  $tp(\bar{a}/N)$  is stationary in the sense of  $T$ .

(For only if part of (2) and (3), we don't need the reduction property.)

PROOF. (1) Let  $\varphi(\bar{x})$  be a formula in  $tp(\bar{a}/M)$ . We show that  $\varphi(\bar{x})$  has a realization in  $N$ . Since  $M \models (\exists \bar{x})(\varphi(\bar{x}) \wedge \bar{x} \in P)$ , we can find a tuple  $\bar{b} \in N$  realizing  $\varphi(\bar{x})$ .

(2) ( $\Rightarrow$ ) Clear.

( $\Leftarrow$ ) Let  $\varphi(\bar{x})$  be a formula in  $tp(\bar{a}/M)$ . By the reduction property, there is an  $L^-(N)$  formula  $\psi(\bar{x})$  such that  $M \models (\forall \bar{x} \in P)[\varphi(\bar{x}) \leftrightarrow \psi^P(\bar{x})]$ . Hence  $\psi(\bar{x}) \in tp^-(\bar{a}/N)$ . And  $\psi(\bar{x}) \in tp^-(\bar{b}/N)$  since  $tp^-(\bar{a}/N) = tp^-(\bar{b}/N)$ . Hence  $\varphi(\bar{x}) \in tp(\bar{b}/M)$ . So  $tp(\bar{a}/M) = tp(\bar{b}/M)$ .

(3) ( $\Rightarrow$ ) We show that  $tp^-(\bar{a}/N\bar{b})$  is an heir over  $N$  in the sense of  $T^-$ . If  $\varphi(\bar{x}\bar{b}) \in tp^-(\bar{a}/N\bar{b})$  and  $\varphi(\bar{x}\bar{y})$  is an  $L^-(N)$ -formula, then  $\varphi^P(\bar{x}\bar{b}) \in tp(\bar{a}/M\bar{b})$ . Since  $\varphi^P(\bar{x}\bar{y})$  is an  $L(M)$ -formula, we can find a tuple  $\bar{b}' \in N$  such that  $\varphi^P(\bar{x}\bar{b}') \in tp(\bar{a}/M\bar{b})$ . Hence  $\varphi(\bar{x}\bar{b}') \in tp^-(\bar{a}/N\bar{b})$ .

( $\Leftarrow$ ) We show that  $tp(\bar{a}/M\bar{b})$  is an heir over  $M$ . If  $\varphi(\bar{x}\bar{b}) \in tp(\bar{a}/M\bar{b})$  and  $\varphi(\bar{x}\bar{y})$  is an  $L(M)$ -formula, then, by the reduction property, there is an  $L^-(N)$ -formula  $\psi(\bar{x}\bar{y})$  such that  $M \models (\forall \bar{x}\bar{y} \in P)[\psi^P(\bar{x}\bar{y}) \leftrightarrow \varphi(\bar{x}\bar{y})]$ . Since  $\psi(\bar{x}\bar{b}) \in tp^-(\bar{a}/N\bar{b})$  and  $\psi(\bar{x}\bar{y})$  is an  $L^-(N)$ -formula, we can find a tuple  $\bar{b}' \in N$  such that  $\psi(\bar{x}\bar{b}') \in tp^-(\bar{a}/N\bar{b})$ . Hence  $\varphi(\bar{x}\bar{b}') \in tp(\bar{a}/M\bar{b})$ .

(4) If  $tp(\bar{a}/N)$  is not stationary in the sense of  $T$ , then we can find tuples  $\bar{b}, \bar{c} \in P^{\mathcal{M}}$  and a model  $M' \supset N$  of  $T$  such that  $tp(\bar{b}/M')$  and  $tp(\bar{c}/M')$  are different non-forking extensions of the type  $tp(\bar{a}/M)$ . By (2) and (3),  $tp^-(\bar{b}/N')$  and  $tp^-(\bar{c}/N')$  are different non-forking extensions of the type  $tp^-(\bar{a}/N)$  in the sense of  $T^-$ . This contradicts the stationarity of  $tp^-(\bar{a}/N)$  in the sense of  $T^-$ .  $\square$

Note. We can see  $\kappa_r(T) \geq \kappa_r(T^-)$  from (3).

### 3. $a$ -models of $T$ and $T^-$

If  $M$  is an  $a$ -model of  $T$ , then  $N$  is an  $a$ -model of  $T^-$  because  $\kappa_r(T) \geq \kappa_r(T^-)$ . The following lemma shows that for any  $a$ -model  $N$  of  $T^-$ , there is an  $a$ -model  $M$  such that  $P^M = N$ .

LEMMA 3.1. *Assume that  $T$  is stable, has the  $\emptyset$ -reduction property over  $P$ , and  $\kappa_r(T) = \kappa_r(T^-)$ . Let  $N$  be an  $a$ -model of  $T^-$ . If  $M'$  is an  $a$ -prime model over  $N$  in the sense of  $T$ , then  $N' = N$ .*

PROOF. If  $N' \neq N$ , we can choose an element  $a \in N' - N$ . Since  $M'$  is  $a$ -prime over  $N$ , there is a subset  $A \subset N$  such that  $|A| < \kappa_r(T)$  and  $\text{stp}(a/A)$  isolates  $tp(a/N)$  in the sense of  $T$ . Let  $\{E_i(xa; \bar{b}_i) : \bar{b}_i \in A, i < \lambda\}$  be an enumeration of  $\text{stp}(a/A)$  where each  $E_i(xy; \bar{b}_i)$  is a finite equivalence relation over  $A$  in the sense of  $T$ . By the  $\emptyset$ -reduction property, for each  $i < \lambda$ , we can find an  $L^-$ -formula  $E'_i(xy; \bar{z}_i)$  such that  $(\forall xy\bar{z}_i \in P)[E_i(xy; \bar{b}_i) \leftrightarrow E'_i(xy; \bar{z}_i)]$  holds in  $M$ .  $E'_i(xy; \bar{b}_i)$  is a finite equivalence relation over  $A$  in the sense of  $T^-$  because so is  $E_i(xy; \bar{b}_i)$  in the sense of  $T$ . Hence  $\{E'_i(xy; \bar{b}_i) : i < \lambda\} \subset \text{stp}^-(a/A)$ . If we choose an element  $c \in N$  as a realization of  $\text{stp}^-(a/A)$ ,  $c$  also realizes  $\text{stp}(a/A)$ , so  $c$  realizes  $tp(a/N)$ . Hence  $tp(a/N) = tp(c/N)$ . But this is a contradiction since  $a \notin N$ .  $\square$

COROLLARY 3.2. *Let  $T$  be stable, minimal over  $P$ , has the  $\emptyset$ -reduction property over  $P$ , and  $\kappa_r(T) = \kappa_r(T^-)$ .*

- (1) *For every  $a$ -model  $N$  of  $T^-$ , there is a unique  $a$ -model  $M$  of  $T$  such that  $P^M = N$ . Moreover,  $|M| = |N|$ .*
- (2) *Every  $a$ -model  $M$  of  $T$  is  $a$ -prime over  $N = P^M$ .*
- (3) *The map which takes  $M$  to  $N = P^M$  is a bijection between  $a$ -models of  $T$  and  $a$ -models of  $T^-$  of the same cardinality.*

When  $T$  is superstable, the number of  $a$ -models is classified by the dimensional order property (definition 4.2) and the deepness (definition 5.4) ([Sh]). We consider these properties in following two sections.

#### 4. Dimensional order property

In this section, we show:

THEOREM 4.1. *If  $T$  is superstable, has the  $\emptyset$ -reduction property over  $P$  and is minimal over  $P$ , then  $T$  has the DOP (dimensional order property) iff  $T^-$  has the DOP. The minimality condition is not necessary for if part.*

First, we recall the definition of the dimensional order property.

DEFINITION 4.2 ([Sh, definition X.2.1]). *let  $T$  be superstable. We say that  $T$  has the dimensional order property (DOP) if there are  $a$ -models  $M_i (i = 0, 1, 2, 3)$  and a regular type  $p \in S(M_3)$  such that:*

- $M_0 < M_1, M_2$ ,
- $M_1$  and  $M_2$  are independent over  $M_0$ ,
- $M_3$  is  $a$ -prime over  $M_1 M_2$ ,
- $p$  is orthogonal to  $M_1$  and  $M_2$ .

To prove the theorem 4.1, we need the following three lemmas.

LEMMA 4.3. *Assume that  $T$  is stable, has the  $\emptyset$ -reduction property over  $P$  and  $\kappa_r(T) = \kappa_r(T^-)$ . Let  $N_0, N_1$  be  $a$ -models,  $N_0 < N_1$ ,  $M_0$   $a$ -prime over  $N_0$ , and  $M_2$  an  $a$ -prime model over  $M_0 N_1$ . Then  $M_2$  is  $a$ -prime over  $N_1$  and  $P^{M_2} = N_1$ .*

PROOF. Let  $M_1$  be an  $a$ -prime model over  $N_1$ . By lemma 3.1,  $P^{M_0} = N_0$  and  $P^{M_1} = N_1$ . We can find an elementary submodel  $M'_0 < M_1$  which is isomorphic to  $M_0$  over  $N_0$  since  $M_0$  is  $a$ -prime over  $N_0$ . By lemma 2.4(1)(4),  $tp(N_1/N_0)$  is stationary and  $tp(N_1/M'_0)$  does not fork over  $N_0$ . Hence  $tp(M'_0/N_1) = tp(M_0/N_1)$ . So we can embed  $M_2$  into  $M_1$  over  $N_1$ . Hence  $M_2$  is  $a$ -prime over  $N_1$  and  $P^{M_2} = N_1$ .  $\square$

LEMMA 4.4. *Assume that  $T$  is stable, has the  $\emptyset$ -reduction property over  $P$  and  $\kappa_r(T) = \kappa_r(T^-)$ . Let  $\bar{a}$  be a tuple from  $P^{\mathcal{M}}$ . Then  $tp(\bar{a}/M)$  is regular in the sense of  $T$  iff  $tp^-(\bar{a}/N)$  is regular in the sense of  $T^-$ .*

PROOF. ( $\Rightarrow$ ) If  $tp^-(\bar{a}/N)$  is not regular in the sense of  $T^-$ , then there is a forking extension  $tp^-(\bar{b}/C)$  which is not orthogonal to  $tp^-(\bar{a}/N)$  in the sense of  $T^-$ . We may assume that

- $tp^-(\bar{a}/C)$  does not fork over  $N$  in the sense of  $T^-$ ,
- $\bar{a}$  and  $\bar{b}$  are dependent over  $C$  in the sense of  $T^-$ .

Let  $N'$  be an  $a$ -model of  $T^-$  such that  $C \subset N'$  and  $tp^-(\bar{a}\bar{b}/N')$  does not fork over  $C$ . By lemma 3.1, there is an  $a$ -model  $M'$  of  $T$  such that  $P^{M'} = N'$ . By lemma 2.4,

- $tp(\bar{a}/M')$  does not fork over  $M$ ,
- $tp(\bar{b}/M')$  is a forking extension of  $tp(\bar{a}/M)$ ,
- $\bar{a}$  and  $\bar{b}$  are dependent over  $M'$ .

Hence  $tp(\bar{a}/M)$  is not regular.

( $\Leftarrow$ ) If  $tp(\bar{a}/M)$  is not regular, there is a forking extension  $tp(\bar{b}/C)$  which is not orthogonal to  $tp(\bar{a}/M)$ . We may assume that:

- $tp(\bar{a}/C)$  does not fork over  $M$ ,
- $\bar{a}$  and  $\bar{b}$  are dependent over  $C$  in the sense of  $T$ .

Let  $M'$  be a model of  $T$  such that  $C \subset M'$  and  $tp(\bar{a}\bar{b}/M')$  does not fork over  $C$ . By lemma 2.4,

- $tp^-(\bar{a}/N')$  does not fork over  $N$  in the sense of  $T^-$ ,
- $tp^-(\bar{b}/N')$  is a forking extension of  $tp^-(\bar{a}/N)$  in the sense of  $T^-$ ,
- $\bar{a}$  and  $\bar{b}$  are dependent over  $N'$  in the sense of  $T^-$ .

Hence  $tp^-(\bar{a}/N)$  is not regular in the sense of  $T^-$ .  $\square$

LEMMA 4.5. *Assume that  $T$  is stable, has the  $\emptyset$ -reduction property over  $P$ . Let  $\bar{a}$  be a tuple from  $P^{\mathcal{M}}$  and  $M_0 < M_1$ .*

- (1) *If  $T$  is minimal over  $P$ ,  $M_0$  and  $M_1$   $a$ -models, and  $tp(\bar{a}/M_1)$  orthogonal to  $M_0$ , then  $tp^-(\bar{a}/N_1)$  is orthogonal to  $N_0$  in the sense of  $T^-$ .*
- (2) *If  $tp^-(\bar{a}/N_1)$  is orthogonal to  $N_0$  in the sense of  $T^-$ , then  $tp(\bar{a}/M_1)$  is orthogonal to  $M_0$ .*

PROOF. (1) If  $tp^-(\bar{a}/N_1)$  is non-orthogonal to  $N_0$  in the sense of  $T^-$ , we can choose a tuple  $\bar{b} \in P^{\mathcal{M}}$  such that:

- $tp^-(\bar{b}/N_1)$  does not fork over  $N_0$  in the sense of  $T^-$ ,
- $\bar{a}$  and  $\bar{b}$  are dependent over  $N_1$  in the sense of  $T^-$ .

By lemma 4.3 and corollary 3.2(1)(2),  $M_1$  is  $a$ -prime over  $M_0N_1$ . Hence, by lemma 2.4(3),

- $tp(\bar{b}/M_1)$  does not fork over  $M_0$ .
- $\bar{a}$  and  $\bar{b}$  are dependent over  $M_1$ .

This shows that  $tp(\bar{a}/M_1)$  is non-orthogonal to  $M_0$ .

- (2) We use the following fact.

FACT 4.6 ([Sh, V.3.4]). *Suppose that  $A \subset B$  and  $p \in S(B)$  is a stationary type. Let  $f$  be an elementary mapping whose domain is  $B$  such that  $f|_A$  is the identity,  $stp(B/A) \equiv stp(f(B)/A)$ , and  $stp(f(B)/B)$  does not fork over  $A$ . Then  $p$  is orthogonal to  $A$  iff  $p$  is orthogonal to  $f(p)$ .*

Let  $f$  be an elementary mapping whose domain is  $M_1$  such that  $f|M_0$  is the identity,  $stp(M_1/M_0) \equiv stp(f(M_1)/M_0)$ , and  $stp(f(M_1)/M_1)$  does not fork over  $M_0$  in the sense of  $T$ . Then, by lemma 2.4,  $f|_{N_1}$  is an elementary mapping whose domain is  $N_1$ ,  $f|_{N_0}$  the identity,  $stp^-(N_1/N_0) \equiv stp^-(f(N_1)/N_0)$ , and  $stp^-(f(N_1)/N_1)$  does not fork over  $N_0$  in the sense of  $T^-$ . By fact 4.6, if  $tp(\bar{a}/M_1)$  is non-orthogonal to  $M_0$ , then  $tp(\bar{a}/M_1)$  is non-orthogonal to



$tp(f(\bar{a})/f(M_1))$ . We may assume that  $tp(\bar{a}/M_2)$  and  $tp(f(\bar{a})/M_2)$  do not fork over  $M_1$  and  $f(M_1)$  respectively,  $\bar{a}$  and  $f(\bar{a})$  are dependent over  $M_2$ , where  $M_2$  is a model containing  $M_1$  and  $f(M_1)$ . By lemma 2.4,  $tp^-(\bar{a}/N_2)$  and  $tp^-(f(\bar{a})/N_2)$  do not fork over  $N_1$  and  $f(N_1)$  respectively,  $\bar{a}$  and  $f(\bar{a})$  are dependent over  $N_2$  in the sense of  $T^-$ . Hence  $tp^-(\bar{a}/N_1)$  is non-orthogonal to  $tp^-(f(\bar{a})/f(N_1))$  in the sense of  $T^-$ . By fact 4.6,  $tp^-(\bar{a}/N_1)$  is non-orthogonal to  $N_0$  in the sense of  $T^-$ .  $\square$

The proof of theorem 4.1 will be completed by following two lemmas.

LEMMA 4.7. *Let  $T$  be superstable, has the  $\emptyset$ -reduction property over  $P$  and is minimal over  $P$ . If  $T$  has the DOP then  $T^-$  has the DOP.*

PROOF. Since  $T$  has the DOP, there are  $a$ -models  $M_i (i = 0, 1, 2, 3)$  and a regular type  $p = tp(\bar{a}/M_3)$  witnessing the conditions for the DOP. By the minimality of  $T$  over  $P$ , we can choose an element  $b \in P^{M_3[\bar{a}]} - N_3$  such that  $tp^-(b/N_3)$  is regular in the sense of  $T^-$  where  $M_3[\bar{a}]$  denotes an  $a$ -prime model over  $M_3\bar{a}$ . We show that  $N_i (i = 0, 1, 2, 3)$  and  $tp(\bar{b}/N_3)$  witness the DOP of  $T^-$ . By lemma 2.4(3),  $N_1$  and  $N_2$  are independent over  $N_0$ .  $N_3$  is  $a$ -prime over  $N_1N_2$ : Let  $N_4$  be an  $a$ -model containing  $N_1$  and  $N_2$  in the sense of  $T^-$ . Let  $M_4$  be an  $a$ -prime model over  $N_4$  in the sense of  $T$ . By lemma 3.1,  $P^{M_4} = N_4$ . By lemma 3.2(2), we can embed  $M_0$  into  $M_4$  over  $N_0$ . By lemma 2.4, this embedding does not change the type of  $N_1N_2M_0$ . Hence we can assume that  $M_0 < M_4$ . Let  $M'_1 < M_1$  be an  $a$ -prime model over  $M_0N_1$ , then  $P^{M'_1} = N_1$ . By the minimality of  $T$  over  $P$ ,  $M_1 = M'_1$ . Hence  $M_1$  is an  $a$ -prime model over  $M_0N_1$ . Similarly,  $M_2$  is an  $a$ -prime model over  $M_0N_2$ . Hence we can embed  $M_1$  and  $M_2$  into  $M_4$  over  $M_0N_1$  and  $M_0N_2$  respectively. By lemma 2.4, this embedding does not change the type of  $M_1M_2$ . Hence we may assume that  $M_1M_2 < M_4$  and can embed  $M_3$  in  $M_4$  over  $M_1M_2$ . Then  $N_3$  is embedded in  $N_4$  over  $N_1N_2$ .

$tp^-(b/N_3)$  is orthogonal to  $N_1$  and  $N_2$  in the sense of  $T^-$ : Since  $tp(\bar{a}/M_3)$  and  $tp(b/M_3)$  are dependent,  $tp(b/M_3)$  is orthogonal to  $M_1$  and  $M_2$ . By lemma 4.5,  $tp^-(b/N_3)$  is orthogonal to  $N_1$  and  $N_2$  in the sense of  $T^-$ .

Hence  $T^-$  has the DOP.  $\square$

LEMMA 4.8. *Let  $T$  be superstable, has the  $\emptyset$ -reduction property over  $P$ . If  $T^-$  has the DOP then  $T$  has the DOP.*

PROOF. Since  $T^-$  has the DOP, there are  $a$ -models  $N_i (i = 0, 1, 2, 3)$  and a regular type  $p = tp^-(\bar{a}/N_3)$  witnessing the conditions for the DOP.

Let  $M_0$  be an  $a$ -prime model over  $N_0$  in the sense of  $T$ . Let  $M_1$  and  $M_2$  be  $a$ -prime models over  $M_0N_1$  and  $M_0N_2$  respectively. By lemma 2.4(3),  $M_1$  and  $M_2$  are independent over  $M_0$ . By lemma 4.3,  $P^{M_i} = N_i (i = 0, 1, 2)$ . Let  $M_4$  be an  $a$ -prime model over  $M_1M_2$ . We may assume that  $N_3 < N_4$  since  $N_3$  is  $a$ -prime over  $N_1N_2$ . Let  $tp^-(\bar{b}/N_4)$  be a non-forking extension of  $p$  in the sense of  $T^-$ . Then  $tp^-(\bar{b}/M_4)$  is orthogonal to  $N_1$  and  $N_2$  in the sense of  $T^-$ . By lemma 4.5,  $q = tp(\bar{b}/N_4)$  is orthogonal to  $M_1$  and  $M_2$ . Hence  $M_i (i = 0, 1, 2, 4)$  and  $q$  witness the conditions for the DOP.  $\square$

By lemmas 4.7 and 4.8, we completed the proof of theorem 4.1.

In lemma 4.7, we assumed the  $\emptyset$ -reduction property and the minimality. In lemma 4.8, we assumed the  $\emptyset$ -reduction property. The following example shows that we can not weaken these assumptions.

EXAMPLE 4.9. (1) *The following example shows that the minimality condition is necessary for lemma 4.7. Let  $E_1(xy)$  and  $E_2(xy)$  be crosscutting equivalence relations where the number of  $E_i$ -classes is infinite ( $i = 1, 2$ ) and each  $E_1$ - $E_2$ -class is infinite. Let  $L = \{P, E_1, E_2\}$  where  $P$  is contained in an  $E_1$ - $E_2$ -class. Let  $L^- = L - \{P\}$ . Since the structure of  $N$  is only equality,  $T$  has the  $\emptyset$ -reduction property and  $T^-$  does not have the DOP. Since each  $E_1$ - $E_2$ -class may have various infinite cardinality,  $T$  has the DOP and is not minimal over  $P$ .*

(2) *The following example shows that the  $\emptyset$ -reduction property is necessary for lemma 4.7. Let  $E_1(xy)$  and  $E_2(xy)$  be crosscutting equivalence relations where the number of  $E_i$ -classes is infinite ( $i = 1, 2$ ) and each  $E_1$ - $E_2$ -class is infinite. Let  $L = \{P, E_1, E_2\}$  where  $P \equiv "x = x"$ . Let  $L^- = \emptyset$ . Then  $T$  is minimal over  $P$  since  $P^M = M$ . But  $T$  does not have the  $\emptyset$ -reduction property over  $P$  and  $T^-$  does not have the DOP since the structure of  $N$  is only equality.  $T$  has the DOP as in (1).*

(3) *The following example shows that the  $\emptyset$ -reduction property is necessary for lemma 4.8. Let  $E_1(xy)$ ,  $E_2(xy)$  and  $E_3(xy)$  be crosscutting equivalence relations where the number of  $E_i$ -classes is infinite ( $i = 1, 2, 3$ ), each  $E_1$ - $E_2$ -class is infinite and each  $E_1$ - $E_2$ - $E_3$ -class is a singleton. Let  $L = \{P, E_1, E_2, E_3\}$  where  $P \equiv "x = x"$ . let  $L^- = \{E_1, E_2\}$ . Then  $T$  does not have the DOP and the  $\emptyset$ -reduction property over  $P$  since the structure of  $M$  is restricted by  $E_3$ .  $T$  is minimal over  $P$  since  $P^M = M$ .  $T^-$  has the DOP as in (1).*

## 5. Deepness

In this section, we show:

**THEOREM 5.1.** *If  $T$  is superstable, has the  $\emptyset$ -reduction property over  $P$  and is minimal over  $P$ , then  $T$  is deep iff  $T^-$  is deep. The minimality condition is not necessary for if part.*

The following definitions are from [Sh].

**DEFINITION 5.2** ([Sh, definition X.1.2]). *For  $A \subset B \subset C$  we say  $B <_A C$  if for every  $\bar{c} \in C$ ,  $tp(\bar{c}/B)$  is orthogonal to  $A$ .*

**DEFINITION 5.3** ([Sh, definition X.4.1]). *Let  $K = \{(M, M', \bar{a}) : tp(\bar{a}/M) \text{ is regular, } M' \text{ is } a\text{-prime over } M\bar{a}, \text{ and } M \text{ is an } a\text{-model}\}$ .*

*For every member of  $K$  we define its depth, an ordinal (zero or successor but not limit) or infinity  $\infty$ , by:*

- (1)  $Dp(M, M', \bar{a}) \geq 0$  iff  $(M, M', \bar{a}) \in K$ ,
- (2)  $Dp(M, M', \bar{a}) \geq \alpha + 1$  ( $\alpha$  zero or successor) iff for some  $M'', \bar{a}'$ :  $(M', M'', \bar{a}') \in K$ ,  $M' <_M M''$  and  $Dp(M', M'', \bar{a}') \geq \alpha$ ,
- (3)  $Dp(M, M', \bar{a}) \geq \delta + 1$  ( $\delta$  limit) iff  $Dp(M, M', \bar{a}) \geq \beta$  for  $\beta < \delta$ ,
- (4)  $Dp(M, M', \bar{a}) = \infty$  iff for every ordinal  $\beta$   $Dp(M, M', \bar{a}) \geq \beta$ ,  
 $Dp(M, M', \bar{a}) = \alpha$  iff  $Dp(M, M', \bar{a}) \geq \alpha$  but not  $Dp(M, M', \bar{a}) \geq \alpha + 1$ .

**DEFINITION 5.4** ([Sh, definition X.4.2]). (1) *The depth of the theory  $Dp(T)$  is  $\sup\{Dp(M, M', \bar{a}) : (M, M', \bar{a}) \in K\}$  when this is finite and  $\sup\{Dp(M, M', \bar{a}) : (M, M', \bar{a}) \in K\} + 1$  when this is infinite.*

(2) *We say the theory  $T$  is deep if its depth is  $\infty$ ; otherwise it is shallow.*

The proof of theorem 5.1 will be completed by following two lemmas.

**LEMMA 5.5.** *Suppose that  $T$  is superstable, has the  $\emptyset$ -reduction property over  $P$  and is minimal over  $P$ . If  $Dp(M, M', \bar{a}) \geq \alpha$ , then there is an element  $b \in N'$  such that  $(N, N', b) \in K$  and  $Dp(N, N', b) \geq \alpha$ .*

**PROOF.** We prove the lemma by the induction on  $\alpha$ .

( $\alpha = 0$ ) If  $Dp(M, M', \bar{a}) \geq 0$  then

- $tp(\bar{a}/M)$  is regular,
- $M'$  is  $a$ -prime over  $M\bar{a}$ ,
- $M$  is  $a$ -model.

$N$  and  $N'$  are  $a$ -models of  $T^-$ . By the minimality of  $T$  over  $P$ , we can choose an element  $b \in N' - N$  such that  $tp^-(b/N)$  is regular in the sense of  $T^-$ .  $tp(b/M)$  is also regular by lemma 4.4. Hence  $M'$  is  $a$ -prime over  $Mb$ . Hence  $N'$  is  $a$ -prime over  $Nb$  in the sense of  $T^-$ : Assume that  $N_0$  is an  $a$ -model containing  $N$  and  $b$ . Let  $M_0$  be an  $a$ -prime model over  $MN_0$ . By lemma 4.3,  $P^{M_0} = N_0$ . Since  $M'$  is  $a$ -prime over  $Mb$ , we can embed  $M'$  into  $M_0$  over  $Mb$ . Hence  $N'$  is embedded in  $N_0$  over  $Nb$ .

( $\alpha = \beta + 1$ ) If  $Dp(M, M', \bar{a}) \geq \beta + 1$ , there are a model  $M''$  and a tuple  $\bar{a}' \in \mathcal{M}$  such that

- $Dp(M', M'', \bar{a}') \in K$ ,
- $M' <_M M''$ ,
- $Dp(M', M'', \bar{a}') \geq \beta$ .

As in the case  $\alpha = 0$ , there is an element  $b \in N'$  such that  $(N, N', b) \in K$  in the sense of  $T^-$ . By the induction hypothesis, there is an element  $b' \in M''$  such that  $(N', N'', b') \in K$  and  $Dp(N, N'', b') \geq \beta$  in the sense of  $T^-$ . Hence it is sufficient to show  $N' <_N N''$  in the sense of  $T^-$ . If not, there are tuples  $\bar{c} \in N''$  such that  $tp^-(\bar{c}/N')$  is non-orthogonal to  $N$  in the sense of  $T^-$ . By lemma 4.5,  $tp(\bar{c}/M')$  is non-orthogonal to  $M$ . This is a contradiction since  $M' <_M M''$ .

( $\alpha = \delta + 1$  where  $\delta$  is limit or  $\alpha = \infty$ ) Clear.  $\square$

**LEMMA 5.6.** *Suppose that  $T$  is superstable, has the  $\emptyset$ -reduction property over  $P$ . If  $Dp(N, N', \bar{a}) \geq \alpha$  in the sense of  $T^-$ , then there are  $a$ -models  $M$  and  $M'$  of  $T$  such that  $P^M = N$ ,  $P^{M'} = N'$  and  $Dp(M, M', \bar{a}) \geq \alpha$ .*

**PROOF.** We prove the lemma by the induction on  $\alpha$ . Let  $M$  be an  $a$ -prime model over  $N$ , and  $M'$  an  $a$ -prime model over  $MN'$ . By lemma 3.1 and lemma 4.3,  $P^M = N$  and  $P^{M'} = N'$ .

( $\alpha = 0$ ) If  $Dp(N, N', \bar{a}) \geq 0$  in the sense of  $T^-$ , then:

- $tp^-(\bar{a}/N)$  is regular in the sense of  $T^-$ ,
- $N'$  is  $a$ -prime over  $N\bar{a}$  in the sense of  $T^-$ ,
- $N$  is an  $a$ -model in the sense of  $T^-$ .

$tp(\bar{a}/M)$  is regular by lemma 4.4. Since  $M'$  is  $a$ -prime over  $MN'$ , it is  $a$ -prime over  $M\bar{a}$ : Assume that  $M_0$  is an  $a$ -model containing  $M$  and  $a$ . We can embed  $N'$  into  $N_0$  over  $Na$ . By lemma 2.4(4), we know that this embedding does not change the type of  $MN'$ . Hence we can embed  $M'$  into  $M_0$  over  $MN'$ .

( $\alpha = \beta + 1$ ) If  $Dp(N, N', \bar{a}) \geq \beta + 1$ , there are a model  $N''$  and a tuple  $\bar{a}' \in P^{\mathcal{M}}$  such that:

- $(N', N'', \bar{a}') \in K$  in the sense of  $T^-$ ,
- $N' <_N N''$  in the sense of  $T^-$ ,
- $Dp(N', N'', \bar{a}') \geq \beta$  in the sense of  $T^-$ .

Let  $M''$  be an  $a$ -prime model over  $M'N''$ . By lemma 4.3,  $P^{M''} = N''$ . As in the case  $\alpha = 0$ , we can show that  $(M', M'', \bar{a}') \in K$ .  $Dp(M', M'', \bar{a}') \geq \beta$  by the induction hypothesis. We show  $M' <_M M''$ . If not, there is a type over  $M$  which is not orthogonal to  $tp(M''/M')$ . Let  $tp(\bar{c}/M')$  be a non-forking extension of the type such that  $\bar{c}$  and  $M''$  are dependent over  $M'$ . Since  $M''$  is  $a$ -prime over  $M'N''$ ,  $\bar{c}$  and  $N''$  are dependent over  $M'$ . Hence  $tp(N''/M')$  is non-orthogonal to  $M$ . By lemma 4.5,  $tp^-(N''/N')$  is non-orthogonal to  $N$  in the sense of  $T^-$ . This is a contradiction since  $N' <_N N''$ .

( $\alpha = \delta + 1$  where  $\delta$  is limit or  $\alpha = \infty$ ) Clear.  $\square$

By lemma 5.5 and lemma 5.6,  $Dp(T) = Dp(T^-)$ . Hence the proof of theorem 5.1 was completed.

## 6. Countable stable theories

In lemma 3.1, we showed that for any  $a$ -model  $N$  of  $T^-$ , there is an  $a$ -model  $M$  of  $T$  such that  $P^M = N$ . In this section, we show that if  $T$  is countable and stable then for any model  $N$  of  $T^-$  there is a model  $M$  of  $T$  with  $P^M = N$ .

**DEFINITION 6.1.** *Let  $A \subset B$ . We say that  $B$  is locally atomic over  $A$  if for any  $\bar{c} \in B$  and a formula  $\varphi(\bar{x}\bar{y})$ , there is a formula  $\psi(\bar{x}) \in p$  such that  $\psi(\bar{x})$  isolates  $p|_{\varphi}$  where  $p = tp(\bar{c}/A)$  and  $p|_{\varphi} = \{\varphi(\bar{x}\bar{b}) : \varphi(\bar{x}\bar{b}) \in p\}$ .*

The following fact is essential for theorem 6.3 below.

**FACT 6.2** ([Sh, IV.3.1]). *Let  $T$  be countable and stable. For any set  $A$ , there is a locally atomic model over  $A$ .*

**THEOREM 6.3.** *Suppose that  $T$  is countable, stable and has the  $\emptyset$ -reduction property. Let  $N$  be a model of  $T^-$  and  $M'$  a locally atomic model over  $N$ . Then  $N' = N$ .*

**PROOF.** If  $N' \neq N$ , then we can choose  $a \in N' - N$ . Since  $M'$  is locally atomic over  $N$ , there is a formula  $\varphi(x\bar{b}) \in p = tp(a/N)$  such that  $\varphi(x\bar{b})$  isolates

$p|_{x \neq y}$  in the sense of  $T$ . By the  $\emptyset$ -reduction property, there is an  $L^-$ -formula  $\psi(x\bar{y})$  such that  $(\forall x\bar{y} \in P)[\varphi(x\bar{y}) \leftrightarrow \psi^P(x\bar{y})]$  holds. Since  $M' \models \varphi(a\bar{b})$ , we can see that  $N' \models (\exists x)\psi(x\bar{b})$ . Hence  $N \models (\exists x)\psi(x\bar{b})$ . Let  $c \in N$  be a witness of  $\psi(x\bar{b})$ . Then  $c$  realizes  $p|_{x \neq y}$  by the choice of  $\varphi$  and  $\psi$ . This is a contradiction since  $x \neq c \in p|_{x \neq y}$ .  $\square$

The following example shows that the countable condition is necessary for theorem 6.3.

**EXAMPLE 6.4.** *There is a stable uncountable theory  $T$  and a model  $N$  of  $T^-$  such that no model  $M$  of  $T$  satisfies  $P^M = N$ .*

Let  $L = \{P, c_i (i < \omega), F_\eta (\eta \in 2^\omega), R_\eta (\eta \in 2^{<\omega})\}$ ,  $L^- = L - \{P\}$  and  $M = \langle M; N, c_i^M (i < \omega), F_\eta^M (\eta \in 2^\omega), R_\eta^M (\eta \in 2^{<\omega}) \rangle$  where

- (i)  $N = \{c_i^M : i < \omega\} \cup \{a\}$ ,
- (ii)  $M = N \cup \{b_\eta : \eta \in 2^\omega\}$ ,
- (iii)  $F_\eta^M$  is a function from  $M - N$  to  $N$ ,
- (iv)  $F_\eta^M(b_v) = c_i^M \Leftrightarrow \eta|_i = v|_i$  and  $\eta(i) \neq v(i)$ ,
- (v)  $F_\eta^M(b_v) = a \Leftrightarrow \eta = v$ ,
- (vi)  $R_\eta^M(b_v) \Leftrightarrow \eta$  is an initial segment of  $v$ .

Then  $T$  is stable and has the  $\emptyset$ -reduction property over  $P$  since any definable set in  $N$  is definable by  $c_i$ s. Let  $N' = \{c_i^M : i < \omega\}$ , then  $N'$  is a model of  $T^-$ . But there is no model  $M'$  such that  $P^{M'} = N'$  because  $tp(a)$  does not have the realization in  $N'$ .

The following example shows that the stability is necessary for theorem 6.3.

**EXAMPLE 6.5.** *There is a countable unstable theory  $T$  and a model  $N$  of  $T^-$  such that no model  $M$  of  $T$  satisfies  $P^M = N$ .*

Let  $L = \{P, R, U_i (i = 1, 2, 3)\}$ ,  $L^- = L - P$  and  $M = \langle M; N, R^M, U_i^M (i = 1, 2, 3) \rangle$  where

- (i)  $M = U_1^M \cup U_2^M \cup U_3^M$  where  $U_3^M$  is the set of all functions from  $U_1^M$  to  $U_2^M$ ,
- (ii)  $N = U_1^M \cup U_2^M$ ,
- (iii)  $R^M(x, y, f) \Leftrightarrow x \in U_1^M \wedge y \in U_2^M \wedge f \in U_3^M \wedge f(x) = y$ ,
- (iv)  $U_1^M$  and  $U_2^M$  are countable.

Then  $T$  is a countable theory with the  $\emptyset$ -reduction property over  $P$  since the structure of  $M$  depends on  $U_3^M$ . Let  $N'$  be a model of  $T^-$  such that  $|U_1^{N'}| \neq |U_2^{N'}|$ . But there is no model  $M'$  such that  $P^{M'} = N'$  because  $|U_1^M| = |U_2^M|$  holds in every model  $M$  of  $T$ .

**References**

- [HP] W. Hodges and A. Pillay, Cohomology of structures and some problems of Ahlbrandt and Ziegler, *J. London Math. Soc.*, **50** (1994), 1–16.
- [KT] H. Kikyo and A. Tsuboi, On reduction property, *J. Symbolic Logic*, **59** (1994), 900–911.
- [Sh] S. Shelah, *Classification Theory*, North-Holland Publ. Co., Amsterdam, 1990.